THE FILLING SCHEME AND THE CHACON–ORNSTEIN THEOREM

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A la mémoire de S. Horowitz, en souvenir de nos discussions

ABSTRACT

A simple proof of the Chacon–Ornstein theorem is derived from a new property of the filling scheme.

Introduction

The "filling scheme" introduced by R. V. Chacon and D. Ornstein in the proof of their celebrated theorem has been extensively used in ergodic theory since and has also been analyzed by H. Rost from the point of view of potential theory.

A property of the sequence $(h_n, n \in N)$ associated by the filling scheme to an arbitrary L^1-function h $[h_0 = h,\ h_{n+1} = Th_n^+ - h_n^-]$ which seems not to have been noticed is that in the conservative ergodic case $\Sigma_N h_n^+ < \infty$ a.e. whenever $\int h < 0$. This property is by no means difficult to prove but its interest lies in the almost immediate proof of the Chacon–Ornstein theorem it gives. The proof so obtained of this theorem rests on no maximal lemma although it may be noticed that maximal lemmas follow easily from the properties of the filling scheme.

1. Preliminaries

Let T be a positive linear transformation of the L^1-space build on a σ-finite measure space $(E, \mathcal{F}, m)$. We suppose that T not only contracts the L^1-norm but that it moreover preserves the integral ($\int Tf = \int f$ for every $f \in L^1$) since we shall mainly be interested in the conservative or recurrent case.

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To each $h \in L^1$, the *filling scheme* associates the sequence $(h_n, n \in N)$ in L^1 which is defined by the recurrence formulas

$$h_0 = h, \quad h_{n+1} = Th_n^+ - h_n^- \qquad (n \in N). \tag{1}$$

The following three properties of this sequence are immediate:

$$h_{n+1}^- \leqq h_n^-, \tag{2}$$

$$h_{n+1}^+ \leqq Th_n^+, \tag{3}$$

$$\int h_{n+1} = \int h_n \qquad (n \in N). \tag{4}$$

Intuitively speaking if f, g are two positive elements of L^1 representing an amount of matter and antimatter respectively laid on E, then letting the antimatter absorb as much as possible of the matter at each point of E, we will be left with amounts of matter and antimatter respectively equal to $(f-g)^+$ and $(g-f)^+$, i.e. h^+ and h^- provided we let $h = f - g$. Next transport the remaining matter by T on E and let the remaining antimatter absorb as much as possible of the transported remaining matter; this second step will leave us with amounts of matter and antimatter respectively equal to $(Th^+ - h^-) = h_1^+$ and $(h^- - Th^+) = h_1^-$. Pursuing this procedure indefinitely, the successive amounts of matter left will be the h_n^+ $(n \in N)$ whose total masses decrease [$\int h_{n+1}^+ \leqq \int h_n^+$ by (3)] and the amounts of antimatter will be the h_n^- $(n \in N)$ which decrease at each point of E [by (2)]. Clearly the positive or negative excess of matter over antimatter remains constant as n varies [property (4)].

The sequence $(h_n, n \in N)$ obtained by the filling scheme is intimately related to the partial sums operators $\Sigma_{k<n} T^k$. This is shown in the next easy lemma which seems not to have appeared in the literature in the following full and useful form.

LEMMA 1. *Given f, $g \in L_+^1$ and letting $h = f - g$, there exist positive elements u_n of L^1 $(n \in N)$ such that the following two formulas hold simultaneously:*

$$\left.\begin{aligned} \sum_{k<n} T^k f &= \sum_{k<n} h_k^+ \\ \sum_{k<n} T^k g &= \sum_{k<n} T^{n-1-k} h_k^- \end{aligned}\right\} + u_n \qquad (n \in N). \tag{5}$$

PROOF. We let by induction

$$u_0 = 0, \quad u_{n+1} = (g - h_n^-) + Tu_n \qquad (n \in N);$$

and prove similarly that the u_n are positive and verify (5). This is clear for $n = 0$ as everything is null. Suppose that our statement is true for n. Then u_{n+1} is positive since $g \geqq h^- \geqq h_n^-$. Moreover since

$$Th_k^+ = h_{k+1} + h_k^- = h_{k+1}^+ + (h_k^- - h_{k+1}^-) \qquad (k \in N)$$

we have by the induction hypothesis

$$T\left(\sum_{k<n} T^k f\right) = \sum_{k<n} h_{k+1}^+ + (h^- - h_n^-) + Tu_n$$

hence

$$\sum_{k<n+1} T^k f = f + \sum_{k<n} h_{k+1}^+ + (h^- - g) + (g - h_n^-) + Tu_n$$

$$= \sum_{k<n+1} h_k^+ + u_{n+1}$$

because $f + (h^- - g) = h^+$. Similarly but more simply

$$T\left(\sum_{k<n} T^k g\right) = \sum_{k<n} T^{n-k} h_k^- + Tu_n$$

by the induction hypothesis; hence

$$\sum_{k<n+1} T^k g = g + \left(\sum_{k<n+1} T^{n-k} h_k^- - h^-\right) + Tu_n$$

$$= \sum_{k<n+1} T^{n-k} h_k^- + u_{n+1}.$$

The induction is thus established and the lemma is proved. ■

The following construction which is equivalent to the filling scheme has some interest, particularly when T is induced by a point transformation. It will however only be used once in the rest of this paper.

LEMMA 2. *Given $h \in L^1$, let $(H_n, n \in N)$ be the sequence defined in L^1 by the following recurrence formulas:*

(6) $$H_0 = h, \quad H_{n+1} = h + TH_n^+ \qquad (n \in N).$$

This sequence is increasing and is related to the sequence $(h_n, n \in N)$ of the filling scheme by the formulas

$$h_n = H_n - H^+_{n-1}$$

(7)

$$h^+_n = H^+_n - H^+_{n-1}, \quad h^-_n = H^-_n \qquad (n \in N)$$

where by convention $H^+_{-1} = 0$.

PROOF. The sequence $(H_n, n \in N)$ increases since $H_1 \geqq h = H_0$ and since $H_{n+1} \geqq H_n$ as soon as $H_n \geqq H_{n-1}$ because the mapping $f \to h + Tf^+$ is increasing. Then we let $h_n = H_n - H^+_{n-1}$ for every $n \in N$ $(h_0 = H_0 = h)$ and show that these h_n satisfy the relations (1) of the filling scheme; however we will first prove that these h_n satisfy all the relations (7).

Either $H_{n-1} \leqq 0$, in which case $h_n = H_n$ so that $h^-_n = H^-_n$, or $H_{n-1} \geqq 0$, in which case $H_n \geqq H_{n-1} \geqq 0$ and $h_n = H_n - H_{n-1} \geqq 0$ so that $H^-_n = 0 = h^-_n$. Hence in all cases $h^-_n = H^-_n$ and $h^+_n = h_n + h^-_n = (H_n - H^+_{n-1}) + H^-_n = H^+_n - H^+_{n-1}$, so that (7) is fully satisfied. Then finally

$$Th^+_n - h^-_n = T(H^+_n - H^+_{n-1}) - H^-_n$$

$$= (H_{n+1} - H_n) - H^-_n = h_{n+1}$$

for all $n \in N$ and since $h_0 = H_0 = h$, the proof is completed. ■

When T commutes with the lattice operations, i.e. essentially when T is of the form $Tf = u\, f \circ \theta$ where θ is a non-singular measurable point transformation of E and $u = d(\theta \circ m)/dm$, the H_n are explicitly given by

$$(8) \qquad H_n = \max[h, h + Th, \cdots, h + \cdots + T^n h] \qquad (n \in N)$$

since then these partial maxima satisfy the relations (6) as is immediately checked (in general it remains true that the H_n are larger than the corresponding partial maxima).

2. A proof of the Chacon–Ornstein theorem

We shall now give a simple property of the filling scheme from which the Chacon–Ornstein theorem immediately follows. For comparison we will simultaneously prove a result due to Rost despite the fact that this result really is very easy, because its proof follows the same line of reasoning (the depth of Rost's complete result lies in the dissipative case which does not concern us here).

To simplify matters we will assume that T is conservative and ergodic, i.e. that $\Sigma_N T^n f = \infty$ for every $f \neq 0$ in L^1_+, in the rest of this section. For a general T, the Hopf's decomposition of the space, which also easily follows from the properties of the filling scheme (see §3), reduces the general case to the conservative ergodic case.

By (2) we may define

$$H^- = \lim_n \downarrow h_n^-. \tag{9}$$

To justify this notation, note that the increasing limit $\lim_n \uparrow H_n$ of the sequence $(H_n, n \in N)$ defined in Lemma 2 which could be denoted by H, has indeed $\lim_n \downarrow h_n^-$ for negative part by property (7). However in this section only H^- and not H will be used.

PROPOSITION 3. *If T is conservative ergodic then for every $h \in L^1$ such that $H^- \neq 0$ one has*

$$\lim_n \downarrow \int h_n^+ = 0 \quad \textit{and} \quad \sum_N h_n^+ < \infty.$$

PROOF. Let $A = \{H^- > 0\}$ and for each $p \in N$, let $v_p = (I_{A^c}T^*)^p 1_A$ where I_B is the multiplication operator by 1_B and where T^* is the dual operator of T acting on L^∞. Then as all probabilists and most ergodicists know, $\Sigma_N v_p = 1$ when T is conservative (i.e. recurrent) and ergodic and when $A \neq \emptyset$.

On the other hand, the inequality $h_{n+1}^+ \leqq Th_n^+$ may be improved to $h_{n+1}^+ \leqq (TI_{A^c})h_n^+$ since $h_n^+ = 0$ on A because $h_n^- \geqq H^- > 0$ there. Hence $h_n^+ \leqq (TI_{A^c})^n h^+$ for every $n \in N$. From this inequality and the definition of the v_p it follows that

$$\int h_n^+ v_p \leqq \int h^+ v_{p+n} \qquad (n, p \in N) \tag{10}$$

since

$$\int h_n^+ \,.\, v_p \leqq \int (TI_{A^c})^n h^+ \cdot v_p = \int h^+ \,.\, (I_{A^c}T^*)^n v_p = \int h^+ v_{p+n}.$$

Summing (10) over the index p gives by the dominated convergence theorem

$$\int h_n^+ = \int h_n^+ \,.\, \sum_p v_p \leqq \int h^+ \sum_p v_{p+n} \downarrow 0$$

as $n \to \infty$. Summing (10) over the index n gives

$$\int \sum_n h_n^+ \,.\, v_p \leqq \int h^+ \sum_n v_{p+n} \leqq \int h^+ < \infty$$

so that the series $\Sigma_n h_n^+$ is finite on each $\{v_p > 0\}$, hence on E since $\Sigma_p v_p > 0$. ∎

To be complete let us briefly recall how the property $\Sigma_N v_p = 1$ is deduced from the hypothesis. First one shows by induction on n that

$$\sum_{p \leq n} v_p + (I_{A^c} T^*)^n 1_{A^c} = 1$$

so that the sequence $\{(I_{A^c} T^*)^n 1_{A^c},\ n \in N\}$ decreases to a positive limit w with the properties $\Sigma_N v_p + w = 1$, $w = 0$ on A and $(I_{A^c} T^*) w = w$. Hence $T^* w \geqq w$. But for any $f \in L^1_+$

$$\int \Big(\sum_{k<n} T^k f \Big) (T^* w - w) = \int f (T^{*n} w - w) \leqq \int f \qquad (n \in N)$$

since w and $T^{*n} w$ are bounded above by 1; as $\Sigma_{k<n} T^k f$ increases to $+\infty$ on E provided $f \not\equiv 0$, this is only possible if $T^* w - w = 0$. Finally choose a $f \not\equiv 0$ in L^1_+ which is zero on A^c, then $\int fw = 0$ and if we write that

$$\int \sum_{k<n} T^k f \cdot w = \int f \cdot \sum_{k<n} T^{*k} w = n \int fw = 0$$

we obtain that $w = 0$ since $\Sigma_{k<n} T^k f \uparrow \infty$ as $n \uparrow \infty$. ■

Let us draw the consequences of the two parts of Proposition 3.

THEOREM 4 (H. Rost). *If T is conservative ergodic, then for every $h \in L^1$ the following two equivalences hold*:

(a) $\lim_n \downarrow \int h_n^+ = 0$ *if and only if* $\int h \leqq 0$,

(b) $\lim_n \downarrow \int h_n^- = 0$, *i.e.* $\lim_n \downarrow h_n^- = 0$ *if and only if* $\int h \geqq 0$.

PROOF. Suppose that $H^- = 0$, i.e. that $\lim_n \downarrow \int h_n^- = 0$, then as a consequence of (4)

$$\int h = \lim_n \int h_n = \lim_n \downarrow \int h_n^+$$

and the two equivalences (a), (b) of the theorem are obvious.

Next suppose that $H^- \not\equiv 0$ so that by the preceding proposition $\lim_n \downarrow \int h_n^+ = 0$; but then (4) implies that

$$\int h = \lim_n \int h_n = -\lim_n \downarrow \int h_n^-$$

and the two equivalences are again obvious. ■

For conservative and ergodic T and for $h = f - g$ $(f, g \in L^1_+)$, this theorem asserts the intuitive fact that the filling scheme finally uses the total amount f of matter, resp. the total amount g of antimatter, if and only if at the start there is more resp. less matter than antimatter. Observe also that the assertion $\Sigma_n h_n^+ < \infty$ under the hypothesis of the lemma says that when $H^- \not\equiv 0$, i.e. when $\int f < \int g$ (see (b) above) the total amount of matter which will pass over x is finite for almost all x.

THEOREM 5 (Chacon–Ornstein). *Let T be conservative ergodic. Then for every $f, g \in L^1_+$, $\lim_{n\to\infty} Q_n(f,g) = \int f / \int g$ a.s. as $n \uparrow \infty$ if $Q_n(f,g)$ denotes the quotient of $\Sigma_{k<n} T^k f$ by $\Sigma_{k<n} T^k g$.*

PROOF. It will be enough to consider the case where $\int f < \int g$. Then if $h = f - g$, the corresponding H^- function differs from the 0 function since otherwise $\int h = \lim_n \int h_n$ would have to be positive; hence by Proposition 3 the series $\Sigma_n h_n^+$ is finite. The formulas (5) then show that

$$\limsup_{n\to\infty} Q_n(f,g) \leqq \limsup_{n\to\infty} \left(\sum_{k<n} h_k^+ \Big/ \sum_{k<n} T^k g \right) + \limsup_{n\to\infty} \left(u_n \Big/ \sum_{k<n} T^k g \right)$$

$$\leqq 0 + 1 = 1$$

since $\Sigma_N T^k g = \infty$ and since $u_n \leqq \Sigma_{k<n} T^k g$ for every $n \in N$.

By applying this result to the pair af, g whenever a is a positive real such that $a \int f < \int g$ and by noting that $Q_n(af,g) = aQ_n(f,g)$, the preceding inequality can be improved to read $\limsup_{n\to\infty} Q_n(f,g) \leqq \int f / \int g$. Finally by interchanging the role of f and g, we have also

$$\liminf_{n\to\infty} Q_n(f,g) = 1 \Big/ \limsup_{n\to\infty} Q_n(g,f) \geqq \int f \Big/ \int g$$

and this already concludes the proof. ■

3. Maximal lemmas

Hopf's maximal lemma has not been used in our proof of the Chacon–Ornstein theorem; it follows however immediately from the general properties of the filling scheme given in §1. The proof so obtained may be considered more "natural" than the beautiful proof found by Garsia [5]. Other easy consequences of the filling scheme are Hopf's decomposition of the space for a general T and the Ornstein inverse maximal lemma for measure preserving point transformations.

The point of departure for all these results is the following consequence of the equality (4) which says that $\int h = \int h_n^+ - \int h_n^-$:

$$\int h \geqq -\int H^- \text{ with equality if } \lim_n \downarrow \int h_n^+ = 0. \tag{11}$$

A. Hopf's maximal lemma states that for every $h \in L^1$

$$\int_E h \geqq 0 \quad \text{if } E = \sup_{n \geqq 1} \left\{ \sum_{k<n} T^k h > 0 \right\}.$$

It follows from (11) and the double inclusion

$$\{h > 0\} \subset E \subset \{H^- = 0\}. \tag{12}$$

The first of these two inclusions is clear. To prove the second inclusion, note that on E where $\Sigma_{k<n} T^k f > \Sigma_{k<n} T^k g$ for at least one $n \in N^+$ ($f = h^+ - g = h^-$), Lemma 1 shows that $h_n^+ > 0$ for some n, hence that $h_n^- = 0$ for the same index; finally $H^- = \lim_p \downarrow h_p^- = 0$ on E. Then to derive Hopf's lemma, one just has to rewrite (11) as

$$\int_E h \geqq \int_{E^c} h^- - \int_{E^c} H^- \geqq 0$$

using the double inclusion above and then the inequality $h^- \geqq H^-$.

B. Hopf's decomposition of the space into a conservative part C and a dissipative part D as well as the definition of the σ-algebra of invariant subsets of C are classically derived from the following relation which holds for any two $f, g \in L_+^1$:

$$g = 0 \quad \text{on} \quad \left\{ \sum_N T^n f = +\infty > \sum_N T^n g \right\}.$$

This relation, which follows from the maximal lemma, is also a consequence of the following intuitively obvious property of the filling scheme.

Lemma. *Let $(f_p, p \in N)$ be a sequence in L_+^1 which decreases to 0 and let $g \in L_+^1$. Then the H^- functions ${}^pH^-$ associated with ${}^ph = f_p - g$ increase to g as $p \uparrow \infty$.*

Proof. Because the mapping $h \to Th^+ - h^-$ is increasing it is easy to see that ${}^pH^-$ increases with p since ${}^ph$ decreases with the parameter. Moreover ${}^pH^- \leqq {}^ph^- \leqq g$ and by (11)

$$\int f_p \geqq \int (g - {}^pH^-).$$

Passing to the limit we obtain that $\lim_p \uparrow {}^pH^- = g$. ■

Given $f, g \in L^1_+$ and applying this lemma to $p^{-1}f$ and g, we obtain that $g = \lim_p \uparrow {}^pH^- = 0$ on the set $\{\Sigma_N T^n f = +\infty > \Sigma_N T^n g\}$ because on this set, by the same argument that led to (11), we have ${}^pH^- = 0$ for every p.

C. When T commutes with the lattice operations, then by letting

$$H = \lim_n \uparrow H_n = \sup_{n \geqq 1} \Big(\sum_{k<n} T^k h \Big) \tag{13}$$

with the H_n defined in Lemma 2, the relation (12) can be immediately improved to

$$E = \{H > 0\}. \tag{14}$$

Let us derive from this equality the inverse maximal lemma proved by Ornstein [7] and Deriennic [4].

LEMMA. *Let θ be a measure preserving transformation which is conservative and ergodic and let T be the corresponding operator on L^1 ($Tf = f \circ \theta$). Then for every $f \in L^1_+$ and every real $a \geqq \int f$*

$$\int_{\{f^* > a\}} f \leqq 2am(f^* > a)$$

if f^* denotes the maximal function: $f^* = \sup_{n \geqq 1}(1/n\, \Sigma_{k<n} T^k f)$.

PROOF. For $h = f - a$, we have $E = \sup_{n \geqq 1}\{\Sigma_{k<n} T^k h > 0\} = \{f^* > a\}$ since $T1 = 1$. On the other hand, Theorem 4 implies that $\lim_n \downarrow \int h_n^+ = 0$ since $\int h < 0$ and T is conservative ergodic; hence $\int h = \lim_n \int h_n = -\int H^-$ and since $\{h > 0\} \subset E = \{H > 0\}$ this can be rewritten

$$\int_E h = \int_{E^c} (h^- - H^-).$$

However since $H = h + (H^+ \circ \theta)$ by the definition of the H_n, we see that $h^- - H^- < h^- \wedge (H^+ \circ \theta) \leqq a\, 1_E \circ \theta$; this implies finally that

$$\int_E (f - a) \leqq \int_{E^c} a\, 1_E \circ \theta \leqq am(E)$$

since m is preserved by θ. ■

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